Question 1. ($20 = 5 + (5 \times 3)$ points)

(a) Fill in the following truth tables for the basic logical operations:

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$	$p \oplus q$
T	T					
T	F					
F	T					
F	F					

(b) Are the following expressions tautologies? For each one, circle "yes" or "no"

yes no $((p \vee q) \wedge (q \vee r)) \rightarrow (p \vee r)$

yes no $((p \rightarrow q) \wedge (p \rightarrow \neg q)) \rightarrow \neg p$

yes no $\neg((p \oplus q) \oplus (q \oplus p))$

yes no $\neg(p \rightarrow q) \leftrightarrow (\neg p \wedge q)$

yes no $((p \rightarrow q) \wedge q) \rightarrow p$

Assume that the universe of discourse is the set of natural numbers $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$. Indicate whether the following statements are true or false by circling the appropriate letter.

T F $\forall x\ (5 \leq x \rightarrow 2 \leq x)$

T F $\exists x\ (10 \leq x \wedge x \leq 8)$

T F $\forall x\ \forall y\ (x \leq y \vee y \leq x)$

T F $\forall x\ \exists y\ (x \cdot x = y)$

T F $\exists x\ \forall y\ (x \cdot y = x)$

T F $\forall x\ \exists y\ (y \cdot y = x)$

T F $\exists x\ \forall y\ (x + y = x)$

T F $\neg \exists x\ \forall y\ (y \leq x)$

T F $\forall x\ \forall y\ (x \leq y \leftrightarrow \exists z\ (x + z = y))$

T F $\forall x\ \exists y\ ((x = 2y) \oplus (x = 2y + 1))$

Question 2. ($18 = 6 \times 3$ points) Let $S(x)$ be the statement "x is a senior", and let $K(x, y)$ be the statement "x knows y", where the universe of discourse is all students. Use quantifiers to express the following statements:

(a) Every student knows John.

(b) All the seniors know each other (including themselves).

(c) Every student knows at least one senior.

(d) Any student who doesn't know him/herself must be a senior.

(e) There are two (different) seniors who don't know each other.

(f) There is exactly one student who doesn't know anyone (i.e., the number of students who don't know anyone is one).

Question 3. ($24 = 4 + 4 + 8 + 6 + 2$ points)

(a) What is the **negation** of the proposition

$$\forall x\ [\exists y\ (C(x) \wedge \neg C(y)) \vee \forall y\ (C(x) \rightarrow C(y))]?$$

Use the De Morgan's laws and other logical identities to simplify your answer so that $\neg$ only is applied to $C(x)$ or $C(y)$.

(b) Suppose that A and B are sets. Show that $A - (B - A) = A$.

(c) Suppose that $A = \{a, b, c\}$ and $B = \{c, d\}$. Give the values of the following sets:

$A \cap B =$ ______________________

$A \cup B =$ ______________________

$A - B =$ ______________________

$A \oplus B =$ ______________________

$A \times B =$ ______________________

The powerset of $B =$ ______________________

(d) Consider the function $f : A \to B$ pictured on the right below:

What is the value of $f(\{2, 3, 4\})$?

What is the value of $f^{-1}(\{c, d\})$?

Is this function one-one? Why?

Is this function onto? Why?

(e) Fill in the blanks with the correct values:

$\lfloor 2.1 \rfloor =$ ______ $\lceil 2.1 \rceil =$ ______ $\lfloor 2.9 \rfloor =$ ______ $\lceil 2.9 \rceil =$ ______

Question 4. ($7 = 2 + 2 + 3$ points)

(a) Express the sum $3 + 5 + 7 + 9 + 11 + \cdots + 27$ using the $\sum$ notation. (You don't need to calculate the answer.)

(b) What is the value of $\sum_{i=2}^{4}(i^2-1)$? Circle your answer.

(c) What is the value of $\sum_{i=2}^{3}\sum_{j=1}^{2}(i-j)$? Circle your answer.

Question 5.

(a) Write down the definition, using quantifiers, of what it means for a function f to be dominated by a function g (i.e., $f \prec g$ or $f = O(g)$).

(b) Use this definition to show that $f(x) = 3x^2 + 2x + 1$ is $O(x^2)$.

(c) Rank the following eight functions of x from 1–8, where 1 is the slowest growing function and 8 is the fastest growing function, by writing the number in the blank next to the function. (For example, if you rank f with a lower number than g, that means that f is dominated by g, i.e., $f \prec g$ or $f = O(g)$.)

3^x ______	$x^2 \cdot x!$ ______	x^2 ______	$(\log x)^{15}$ ______
$x^2(\log x)^5$ ______	x^x ______	$x^6 \cdot 2^x$ ______	x^3 ______
x^4 ______	x^x ______	4^x ______	$x^2 \cdot 3^x$ ______
x^3 ______	$2^x \cdot x!$ ______	$(\log x)^5$ ______	$x^3(\log x)^2$ ______

Question 6. ($8 = 4 \times 2$ points) Integers and divisibility.

(a) List all the prime numbers between 50 and 60.

(b) List all of the positive integers x such that $3 \mid x$ and $x \mid 24$.

(c) What is the smallest integer x such that $x \geq 30$ and $x \equiv 11 \pmod 8$? ______

(d) What is $\gcd(2^5 \cdot 3^7 \cdot 5^2 \cdot 2^2 \cdot 5^4 \cdot 7^2)$? ____________________

Question 7. (5 points) Use the Euclidean Algorithm to find the value of $\gcd(192, 162)$. Show your work, and circle your answer.

Question 8. ($8 = 2 + 3 + 3$ points) Matrix operations

(a) What is $\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$?

What is $\begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$?

What is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$?

What is $\begin{bmatrix} 5 & -2 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ -3 & 3 \end{bmatrix}$?

What is $2 \cdot \begin{bmatrix} 5 & -2 \\ 0 & 2 \end{bmatrix}$?

What is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}$?

Question 2. (15 = 5 × 3 points) Let $S(x)$ be the statement "x is a senior", and let $K(x, y)$ be the statement "x knows y", where the universe of discourse is all students. Use quantifiers to express the following statements:

(a) Every senior knows Jeffrey.

(b) No one knows everyone.

(c) Everybody knows somebody.

(d) John knows at least two seniors.

(e) Jim is the only student who doesn't know himself.

Question 3. (24 = 8 + 4 + 12 points)

(a) Suppose that $A = \{1, 2, 3\}$ and $B = \{2, 4\}$. Give the values of the following sets:

$A \cap B =$ ______________________________

$A \cup B =$ ______________________________

$A - B =$ ______________________________

$A \oplus B =$ ______________________________

$A \times B =$ ______________________________

The powerset of $B =$ ______________________________

(b) Suppose $X = \{a, b, c\}$ and $Y = \{b, \{c\}\}$. Indicate whether the following statements are true or false by circling the appropriate letter.

T F $c \in X - Y$.

T F $Y \subseteq X$.

T F $\{c\} \subseteq Y$.

T F $\{\} \subseteq X \times X$.

$f : \mathbf{N} \to \mathbf{N}$	one-to-one	onto	$f(\{1,2\})$	$f^{-1}(\{1,2\})$
$f(x) = 2x + 3$	yes no	yes no		
$f(x) = \lfloor (x+1)/2 \rfloor$	yes no	yes no		
$f(x) = x$	yes no	yes no		

Question 4. (8 = 2 + 3 + 3 points)

(a) Express the sum $5 + 11 + 17 + 23 + 29 + \cdots + 65$ using the $\sum$ notation. (You don't need to calculate the answer.)

(b) What is the value of $\displaystyle\sum_{i=2}^{3}\sum_{j=1}^{2}(i+j)$? Circle your answer.

(c) What is the value of $\displaystyle\sum_{i=0}^{40} 3i + 1$? Circle your answer

Question 6. (6 = 3 × 2 points) Integers and divisibility.

(a) List all the prime numbers between 40 and 50.

(b) List three positive integers x such that $(3 \mid x) \wedge (5 \mid x) \wedge \neg(2 \mid x)$.

(c) What is the smallest positive integer x such that $x + 1 \equiv 2x \pmod{11}$?

Question 7. (6 points) Use the Euclidean Algorithm to find the value of $\gcd(84, 45)$. Show each step, and circle your final answer.